A Functional Equation for the Spectral Fourth Moment of Modular Hecke L-Functions

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Albeit essential corrections are required both in his claim and in his argument, N.V. Kuznetsov observed in [2] a highly interesting transformation formula for spectral sums of products of four values of modular Hecke L-functions. A complete proof of a corrected version of the formula was later supplied by the present author [3], which has, however, remained unpublished for more than a decade, except for a limited distribution. The aim of the present paper is to reproduce [3] with some sophistications, retaining the original style as much as possible.

Convention: Notations become available at their first appearances and will continue to be effective thereafter.

1. We shall first introduce basic notions; for details we refer to [4]. Thus, let Γ be the full modular group $\mathrm{PSL}_2(\mathbb{Z})$, and $L^2(\Gamma\backslash\mathbb{H})$ the Hilbert space composed of all Γ-automorphic functions on $\mathbb{H}=\{x+iy : y>0\}$, which are square integrable over $\Gamma\backslash\mathbb{H}$ against the hyperbolic measure. A real analytic modular cusp form or a Maass form is an element in $L^2(\Gamma\backslash\mathbb{H})$, which is an eigenfunction of the hyperbolic Laplacian $\Delta=-y^2((\partial/\partial x)^2+(\partial/\partial y)^2)$. The subspace spanned by all Maass forms has a complete orthonormal system $\{\psi_j : j=1,2,\cdots\}$ such that $\Delta\psi_j=(\frac14+\kappa_j^2)\psi_j$ with $0<\kappa_1\le\kappa_2\le\cdots$, and

$$\psi_j(x+iy)=\sqrt{y}\sum_{n\ne0}\varrho_j(n)K_{i\kappa_j}(2\pi|n|y)e(nx),\quad x+iy\in\mathbb{H}, \tag{1.1}$$

where K_ν is the K-Bessel function of order ν, and $e(x)=\exp(2\pi ix)$. The $\varrho_j(n)$ are called the Fourier coefficients of ψ_j. We may assume as usual that ψ_j are simultaneous eigenfunctions of all Hecke operators with corresponding eigenvalues $t_j(n)\in\mathbb{R}$; that is, for each integer $n>0$

$$\frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b \bmod d}\psi_j((az+b)/d)=t_j(n)\psi_j(z),\quad z\in\mathbb{H}, \tag{1.2}$$

and that

$$\psi_j(-\overline{z})=\epsilon_j\psi_j(z),\quad \epsilon_j=\pm1. \tag{1.3}$$

We have, uniformly in j,

$$t_j(n)\ll n^{\frac14+\varepsilon} \tag{1.4}$$

for any fixed $\varepsilon > 0$. The Hecke L-function associated with ψ_j is defined by

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s}, \quad \operatorname{Re} s > 1. \tag{1.5}$$

This continues to an entire function satisfying a Riemannian functional equation; in consequence, $H_j(s)$ is of polynomial growth with respect to both κ_j and s in any fixed vertical strip.

We need also to define holomorphic modular cusp forms, and corresponding Hecke L-functions. Thus, if ψ is holomorphic over $\mathbb{H}$, and $\psi(z)(dz)^k$ with an integer $k > 0$ is Γ invariant, then ψ is a holomorphic modular form of weight $2k$. If ψ vanishes at the point infinity, then it is called a cusp form. The set composed of all such functions is a finite dimensional Hilbert space. Let $\{\psi_{j,k} : 1 \le j \le \vartheta(k)\}$ be an orthonormal basis. The Fourier coefficients $\varrho_{j,k}(n)$ of $\psi_{j,k}$ is defined by the expansion

$$\psi_{j,k}(z) = \sum_{n=1}^{\infty} n^{k-\frac{1}{2}} \varrho_{j,k}(n) e(nz), \quad z \in \mathbb{H}. \tag{1.6}$$

We may assume again that $\psi_{j,k}$ are simultaneous eigenfunctions of all Hecke operators, so that there exist real numbers $t_{j,k}(n)$ such that

$$\frac{1}{\sqrt{n}} \sum_{ad=n} (a/d)^k \sum_{b \bmod d} \psi_{j,k}((az+b)/d) = t_{j,k}(n) \psi_{j,k}(z). \tag{1.7}$$

Corresponding to (1.4), we have, uniformly in j, k,

$$t_{j,k}(n) \ll n^{\frac{1}{4}+\varepsilon}. \tag{1.8}$$

Also the Hecke L-function attached to $\psi_{j,k}$ is defined by

$$H_{j,k}(s) = \sum_{n=1}^{\infty} t_{j,k}(n) n^{-s}, \quad \operatorname{Re} s > 1, \tag{1.9}$$

which continues to an entire function of polynomial growth with respect to both k and s in any fixed vertical strip.

2. Now, let $h(r)$ be even, entire, and

$$h(r) \ll \exp(-c|r|^2). \tag{2.1}$$

This assumption is made for the sake of simplicity; in fact, we can relax it considerably. For instance, our argument works well, provided that $h(r)$ is regular and $\ll (|r|+1)^{-A}$ for $|\operatorname{Im} r| < A$ with a sufficiently large A.

In the region of absolute convergence, i.e.,

$$z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4, \quad \operatorname{Re} z_j > 1 \quad (1 \le j \le 4), \tag{2.2}$$

let

$$\mathcal{H}_\pm(z; h) = \sum_{j=1}^{\infty} \alpha_j \epsilon_j^{\frac{1}{2}(1\mp 1)} H_j(z_1) H_j(z_2) H_j(z_3) H_j(z_4) h(\kappa_j) \tag{2.3}$$

with $\alpha_j = |\varrho_j(1)|^2 / \cosh(\pi\kappa_j)$, and

$$\begin{aligned} \mathcal{C}(z; h) = \frac{1}{\pi} \int_{-\infty}^{\infty} & \zeta(z_1 + ir)\zeta(z_2 + ir)\zeta(z_3 + ir)\zeta(z_4 + ir) \\ \times\, \zeta(z_1 - ir)&\zeta(z_2 - ir)\zeta(z_3 - ir)\zeta(z_4 - ir) \frac{h(r)}{\zeta(1 + 2ir)\zeta(1 - 2ir)} dr \end{aligned} \tag{2.4}$$

with ζ the Riemann zeta-function. Put

$$\mathcal{G}(z; h) = (\mathcal{H}_+ + \mathcal{C})(z; h). \tag{2.5}$$

Obviously $\mathcal{H}_+(z; h)$ is entire over $\mathbb{C}^4$. On the other hand, $\mathcal{C}(z; h)$ is meromorphic, which can be seen by shifting the contour vertically. Thus, $\mathcal{G}(z; h)$ is meromorphic over $\mathbb{C}^4$. Interesting situations occur when z is in the set

$$\mathcal{X} = \left(\tfrac{1}{2} + i\mathbb{R}\right)^4. \tag{2.6}$$

or in its vicinity. Then we have

$$\mathcal{G}(z; h) = (\mathcal{H}_+ + \mathcal{C}_0 + \mathcal{R}_0)(z; h), \tag{2.7}$$

where $\mathcal{C}_0$ has the same integral representation as $\mathcal{C}$ but with the present specification for z, and

$$\begin{aligned} &\mathcal{R}_0(z; h) \\ &= 4\zeta(z_2 - z_1 + 1)\zeta(z_3 - z_1 + 1)\zeta(z_4 - z_1 + 1) \\ &\quad \times \zeta(z_2 + z_1 - 1)\zeta(z_3 + z_1 - 1)\zeta(z_4 + z_1 - 1) h(i(z_1 - 1))/\zeta(3 - 2z_1) \\ &+ 4\zeta(z_1 - z_2 + 1)\zeta(z_3 - z_2 + 1)\zeta(z_4 - z_2 + 1) \\ &\quad \times \zeta(z_1 + z_2 - 1)\zeta(z_3 + z_2 - 1)\zeta(z_4 + z_2 - 1) h(i(z_2 - 1))/\zeta(3 - 2z_2) \\ &+ 4\zeta(z_1 - z_3 + 1)\zeta(z_2 - z_3 + 1)\zeta(z_4 - z_3 + 1) \\ &\quad \times \zeta(z_1 + z_3 - 1)\zeta(z_2 + z_3 - 1)\zeta(z_4 + z_3 - 1) h(i(z_3 - 1))/\zeta(3 - 2z_3) \\ &+ 4\zeta(z_1 - z_4 + 1)\zeta(z_2 - z_4 + 1)\zeta(z_3 - z_4 + 1) \\ &\quad \times \zeta(z_1 + z_4 - 1)\zeta(z_2 + z_4 - 1)\zeta(z_3 + z_4 - 1) h(i(z_4 - 1))/\zeta(3 - 2z_4) \end{aligned} \tag{2.8}$$

(cf. pp. 117–118 of [4]). It can be shown, with an elementary argument, that $\mathcal{R}_0$ is finite on $\mathcal{X}$.

Remark: Both [2] and [3] treat $\mathcal{G}$. One may include $\mathcal{H}_-$ into discussion to make the subject more symmetric: The sum

$$\mathcal{H}_+(z;h_+)+\mathcal{H}_-(z;h_-)+\mathcal{C}(z;h_++h_-) \tag{2.9}$$

could be chosen to start with. To attain a fuller symmetry, the sum $\mathcal{H}_0$, defined by (5.20) below, is also to be included. We note that dealing with a spectral sum involving $H_j(\frac{1}{2})$ one may exploit the fact that $H_j(\frac{1}{2})=\epsilon_j H(\frac{1}{2})$ (see Section 3.3 of [4]), and that $\mathcal{H}_-$ is much less problematic than $\mathcal{H}_+$. Hence, sometimes, $\mathcal{H}_-$ can be a better choice than $\mathcal{H}_+$ to work with. These refinements are nevertheless omitted in our present discussion, because of their peripheral nature.

Returning to the domain (2.2), we have

$$\begin{aligned}\mathcal{H}_+(z;h) &= \zeta(z_1+z_2)\zeta(z_3+z_4)\\ &\times \sum_{m,n\ge1}\frac{\sigma_{z_1-z_2}(m)\sigma_{z_3-z_4}(n)}{m^{z_1}n^{z_3}}\sum_{j\ge1}\alpha_j t_j(m)t_j(n)h(\kappa_j),\end{aligned} \tag{2.10}$$

by the multiplicativity of Hecke eigenvalues (see (3.2.7) of [4]). To the inner sum we apply the Spectral-Kloosterman sum formula due to Bruggeman and Kuznetsov (Theorems 2.2 of [4]). We have

$$\begin{aligned}\mathcal{G}(z;h) &= \frac{1}{\pi^2\zeta(z_1+z_2+z_3+z_4)}\\ &\times \zeta(z_2+z_3)\zeta(z_2+z_4)\zeta(z_3+z_4)\int_{-\infty}^{\infty} rh(r)\tanh\pi r\,dr\\ &+\zeta(z_1+z_2)\zeta(z_3+z_4)\sum_{m,n\ge1}\frac{\sigma_{z_1-z_2}(m)\sigma_{z_3-z_4}(n)}{m^{z_1}n^{z_3}}\mathcal{K}(m,n;\varphi).\end{aligned} \tag{2.11}$$

Here

$$\mathcal{K}(a,b;\varphi)=\sum_{c\ge1}\frac{1}{c}S(a,b;c)\varphi\left(\frac{4\pi}{l}\sqrt{|ab|}\right), \tag{2.12}$$

where $S(m,n;l)$ is the Kloosterman sum, and, for $x>0$,

$$\varphi(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty}(J_{-2ir}(x)-J_{2ir}(x))\frac{rh(r)}{\cosh\pi r}dr, \tag{2.13}$$

with J_ν the J-Bessel function of order ν. Considering the Mellin transform of φ, we have

$$\varphi(x)=\frac{1}{\pi^2}\int_{(\alpha)}h^*(s)(x/2)^{-2s}ds,\quad -\frac{1}{2}<\alpha<0, \tag{2.14}$$

where (α) is the vertical line $\mathrm{Re}\, s = \alpha$, and

$$h^*(s) = \int_{-\infty}^{\infty} \frac{\Gamma(s+ir)}{\Gamma(1-s+ir)} \frac{rh(r)}{\cosh \pi r} dr. \tag{2.15}$$

A downward shift of the contour shows that $h^*(s)$ is regular for $\mathrm{Re}\, s > -\frac{1}{2}$, and there

$$h^*(s) \ll (|s|+1)^{2\mathrm{Re}\, s-1}, \tag{2.16}$$

which is in fact the best possible. However, this bound is not adequate to carry out subsequent transformations of $\mathcal{G}$, a grave fact which is overlooked in [2].

On the other hand the corresponding transform of h attached to $\mathcal{H}_-$ is

$$\int_{-\infty}^{\infty} \frac{\Gamma(s+ir)}{\Gamma(1-s+ir)} \frac{rh(r)}{\cos \pi s} dr. \tag{2.17}$$

This decays exponentially, which makes $\mathcal{H}_-$ much easier to treat than $\mathcal{H}_+$. See also p. 113 of [4].

3. To overcome this difficulty with h^*, we appeal to a device that has stemmed from an idea of R. Murty, which exploits the fact that the full modular group lacks holomorphic cusp forms of low weights (see pp. 442–443 of [5] and Concluding Remark below).

Put

$$h^{**}(s) = h^*(s) + \frac{1}{2}\pi i \sum_{\substack{k=1 \\ k\neq 6}}^{7} \gamma_k \frac{\Gamma(k-\frac{1}{2}+s)}{\Gamma(k+\frac{1}{2}-s)}, \tag{3.1}$$

where for $1 \le k \le 3$

$$\gamma_k = (-1)^k \frac{2}{\pi}(2k-1)h(i(k-\tfrac{1}{2})), \tag{3.2}$$

and other γ_k are to satisfy, for $\nu = 0, 1, 2$,

$$\sum_{\substack{k=1 \\ k\neq 6}}^{7} (-1)^k \gamma_k (k-\tfrac{1}{2})^{2\nu} = (-1)^\nu \frac{2}{\pi} \int_{-\infty}^{\infty} r^{2\nu+1} h(r) \tanh \pi r\, dr. \tag{3.3}$$

Obviously the γ_k are fixed uniquely. With this, $h^{**}(s)$ is regular over $\mathbb{C}$ except for the simple poles at $s = -(k-\frac{1}{2})$ $(k = 4, 5, \ldots)$. Moreover, as s tends to infinity in any fixed vertical strip, we have

$$h^{**}(s) \ll |s|^{2\mathrm{Re}\, s-4}. \tag{3.4}$$

The first assertion is obtained by shifting the contour in (2.15) to $\mathrm{Im}\, r = -3$. The second will be proved later for the sake of completeness. We remark

that the modification (3.1) does not cause much changes in (2.11), at least outwardly. In fact, we have

$$\mathcal{G}(z;h) = \zeta(z_1+z_2)\zeta(z_3+z_4)\sum_{m,n\ge 1}\frac{\sigma_{z_1-z_2}(m)\sigma_{z_3-z_4}(n)}{m^{z_1}n^{z_3}}\mathcal{K}(m,n;\tilde\varphi), \tag{3.5}$$

with

$$\tilde\varphi(x) = \frac{1}{\pi^2}\int_{(\alpha)} h^{**}(s)(x/2)^{-2s}ds. \tag{3.6}$$

To see this, note that (3.1) implies

$$\tilde\varphi(x) = \varphi(x) - \sum_{\substack{k=1\\ k\ne 6}}^{7}\gamma_k J_{2k-1}(x), \tag{3.7}$$

and that for $k = 1, 2, 3, 4, 5, 7$

$$\begin{aligned} q_{m,n}(k) &= \sum_{l\ge 1}\frac{1}{l}S(m,n;l)J_{2k-1}\left(\frac{4\pi}{l}\sqrt{mn}\right)\\ &= \delta_{m,n}\frac{(-1)^{k-1}}{2\pi}, \end{aligned} \tag{3.8}$$

which is equivalent to the fact $\vartheta(k) = 0$ for these values of k (see Lemma 2.3 of [4]). The last two identities and (3.3), $\nu = 0$, give

$$\mathcal{K}(m,n;\tilde\varphi) = \mathcal{K}(m,n;\varphi) + \frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^{\infty} rh(r)\tanh \pi r\, dr \tag{3.9}$$

Inserting this into (3.5), we recover (2.11).

To prove (3.4), we introduce the contour $\mathcal{L}$ which is the result of connecting, with straight lines, the points $-\infty - ia$, $-b - ia$, $b + ia$, $+\infty + ia$ in this order, where $a, b > 0$ are arbitrary but fixed. The analytic continuation of $h^{**}(s)$ to the region on the left of $-i\mathcal{L}$ is given by

$$\begin{aligned} h^{**}(s) &= \frac{\cos\pi s}{i\pi}\int_{\mathcal{L}} rh(r)\tanh\pi r\,\Gamma(s+ir)\Gamma(s-ir)dr\\ &\quad - \frac{\cos\pi s}{2i}\sum_{\substack{k=1\\ k\ne 6}}^{7}(-1)^k\gamma_k\Gamma(k-\tfrac12+s)\Gamma(\tfrac12-k+s). \end{aligned} \tag{3.10}$$

Let us suppose that s is on the left of $-i\mathcal{L}$, and $t = \operatorname{Im} s$ tends to $+\infty$ while $-a < \sigma = \operatorname{Re} s$ is bounded. Since a is arbitrary, this is essentially the same

as to assume only that t tends to $+\infty$ while σ remains bounded. Then an appropriate deform of $\mathfrak{L}$ gives

$$\begin{aligned} h^{**}(s) = \frac{\cos(\pi s)}{i\pi} &\int_{-\log t}^{\log t} rh(r)\tanh \pi r\, \Gamma(s+ir)\Gamma(s-ir)dr \\ &- \frac{\cos(\pi s)}{2i}\sum_{\substack{k=1\\k\neq 6}}^{7}(-1)^k\gamma_k\Gamma(k-\tfrac{1}{2}+s)\Gamma(\tfrac{1}{2}-k+s) \\ &+ O_\sigma\left(\exp(-c(\log t)^2)\right), \end{aligned} \tag{3.11}$$

because of (2.1). In this integral, we have, by Stirling's formula,

$$\begin{aligned} &\Gamma(s+ir)\Gamma(s-ir) = -2\pi i t^{2\sigma-1+2it}\exp(\pi i\sigma - 2it - \pi t) \\ &\times \left(1 + t^{-1}p_1(\sigma, r^2) + t^{-2}p_2(\sigma, r^2) + O_\sigma((|r|+1)^6 t^{-3})\right), \end{aligned} \tag{3.12}$$

where the polynomial $p_\nu(\sigma, \xi)$ is of degree ν in ξ. Thus the first term on the right of (3.11) is equal to

$$\begin{aligned} &-t^{2\sigma-1+2it}e^{-2it}\int_{-\infty}^{\infty} rh(r)\tanh \pi r \\ &\quad\times \left(1 + t^{-1}p_1(\sigma, r^2) + t^{-2}p_2(\sigma, r^2)\right)dr + O_\sigma(t^{2\sigma-4}). \end{aligned} \tag{3.13}$$

By (3.3), we may put this as

$$\begin{aligned} -\frac{\pi}{2}t^{2\sigma-1+2it}e^{-2it}&\sum_{\substack{k=1\\k\neq 6}}^{7}(-1)^k\gamma_k\big(1 + t^{-1}p_1(\sigma, (i(k-\tfrac{1}{2}))^2) \\ &\quad + t^{-2}p_2(\sigma, (i(k-\tfrac{1}{2}))^2)\big) + O_\sigma(t^{2\sigma-4}), \end{aligned} \tag{3.14}$$

which is, by (3.12), equal to

$$\frac{1}{2i}\sum_{\substack{k=1\\k\neq 6}}^{7}(-1)^k\gamma_k \cos \pi s\, \Gamma(k-\tfrac{1}{2}+s)\Gamma(\tfrac{1}{2}-k+s) + O_\sigma(t^{2\sigma-4}). \tag{3.15}$$

We end the proof for the case $\mathrm{Im}\, s > 0$. The case $\mathrm{Im}\, s < 0$ is analogous.

4. We may now return to (3.5). We are about to transform it with Estermann's functional equation for

$$D(s, \xi; e(a/l)) = \sum_{n\geq 1} n^{-s}\sigma_\xi(n)e(an/l), \quad (a, l) = 1, \tag{4.1}$$

basic properties of which are given in Lemma 3.7 of [4]. To this end, let us assume that

$$\mathrm{Re}\, z_j > 1 - \alpha \quad (1 \le j \le 4), \quad -\frac{5}{2} < \alpha < -\frac{1}{2}. \tag{4.2}$$

Then we see readily, from (3.5)–(3.6), that

$$\begin{aligned} \mathcal{G}(z;h) = \frac{1}{\pi^2}\zeta(z_1+z_2)\zeta(z_3+z_4) \sum_{l\ge 1} \frac{1}{l} \sum_{\substack{a \bmod l \\ (a,l)=1}} \int_{(\alpha)} (2\pi/l)^{-2s} h^{**}(s) \\ \times D(z_1+s; z_1-z_2; e(a/l)) D(z_3+s, z_3-z_4; e(a^*/l)) ds, \end{aligned} \tag{4.3}$$

with $aa^* \equiv 1 \bmod l$, which converges absolutely. We consider the following subdomain of (4.2):

$$1 - \alpha < \mathrm{Re}\, z_j < -\beta \quad (1 \le j \le 4), \quad -\frac{7}{2} < \beta < \alpha - 1. \tag{4.4}$$

Also we introduce the domain

$$\mathrm{Re}\,(z_1+z_2+z_3+z_4) > 3 - 2\beta. \tag{4.5}$$

We shall work temporarily in the intersection of these two domains, which is not empty with an appropriate choice of α, β.

We then shift the contour in (4.3) to $\mathrm{Re}\, s = \beta$, which can be performed because of (3.4) and the convexity bound for $D(s,\xi;e(a/l))$. We encounter poles at $s = 1 - z_j$ $(1 \le j \le 4)$, which can be assumed, without loss of generality, are all simple. Computing the residue, we get

$$\mathcal{G}(z;h) = (\mathcal{R}_1 + \mathcal{G}_1)\,(z;h). \tag{4.6}$$

Here $\mathcal{G}_1$ has the same expression as the right side of (4.3) but with the contour (β), and

$$\mathcal{R}_1(z;h) = \frac{2i}{\pi}\zeta(z_1+z_2)\zeta(z_3+z_4)\sum_{l\ge 1}\frac{1}{l}\sum_{\substack{a \bmod l \\ (a,l)=1}}\{\cdots\}, \tag{4.7}$$

with

$$\begin{aligned} \{\cdots\} =& (2\pi/l)^{2(z_1-1)}\zeta(z_2-z_1+1) l^{z_1-z_2-1} \\ &\quad \times D(z_3-z_1+1, z_3-z_4; e(a^*/l)) h^{**}(1-z_1) \\ &+ (2\pi/l)^{2(z_2-1)}\zeta(z_1-z_2+1) l^{z_2-z_1-1} \\ &\quad \times D(z_3-z_2+1, z_3-z_4; e(a^*/l)) h^{**}(1-z_2) \\ &+ (2\pi/l)^{2(z_3-1)}\zeta(z_4-z_3+1) l^{z_3-z_4-1} \\ &\quad \times D(z_1-z_3+1, z_1-z_2; e(a/l)) h^{**}(1-z_3) \\ &+ (2\pi/l)^{2(z_4-1)}\zeta(z_3-z_4+1) l^{z_4-z_3-1} \\ &\quad \times D(z_1-z_4+1, z_1-z_2; e(a/l)) h^{**}(1-z_4). \end{aligned} \tag{4.8}$$

The $\mathcal{R}_1(z;h)$ can be expressed in terms of the Riemann zeta-function. To see this we assume temporarily that z_j are close to each other, while satisfying (4.4)–(4.5) and $z_j \neq z_{j'}$ for $j \neq j'$. Then the right side of (4.7) obviously converges to a regular function of z in such a domain. With this, let us further suppose that $\operatorname{Re} z_1 < \operatorname{Re} z_3$, $\operatorname{Re} z_1 < \operatorname{Re} z_4$. We have, because of absolute convergence,

$$\begin{aligned}
&\sum_{l\geq 1} l^{-z_1-z_2} \sum_{\substack{a \bmod l \\ (a,l)=1}} D(z_3-z_1+1, z_3-z_4; e(a/l)) \\
=&\sum_{l\geq 1} l^{-z_1-z_2} \sum_{n\geq 1} c_l(n)\sigma_{z_3-z_4}(n) n^{z_1-z_3-1} \\
=&\sum_{n\geq 1} \sigma_{z_3-z_4}(n) n^{z_1-z_3-1} \sum_{l\geq 1} c_l(n) l^{-z_1-z_2} \\
=&\frac{1}{\zeta(z_1+z_2)} \sum_{n\geq 1} \sigma_{1-z_1-z_2}(n)\sigma_{z_3-z_4}(n) n^{z_1-z_3-1} \\
=&\frac{\zeta(z_3-z_1+1)\zeta(z_4-z_1+1)\zeta(z_2+z_4)\zeta(z_2+z_4)}{\zeta(z_1+z_2)\zeta(z_3-z_1+z_3+z_4+1)},
\end{aligned} \tag{4.9}$$

where $c_l(n)$ is the Ramanujan sum. The last expression gives a meromorphic continuation of the initial sum to $\mathbb{C}^4$. In this way we find that $\mathcal{R}_1$ exists as a meromorphic function over $\mathbb{C}^4$, and admits the expression

$$\begin{aligned}
&\mathcal{R}_1(z;h) \\
=&i\frac{(2\pi)^{2z_1-1}h^{**}(1-z_1)}{\pi^2\zeta(1-z_1+z_2+z_3+z_4)}\zeta(1-z_1+z_2)\zeta(1-z_1+z_3)\zeta(1-z_1+z_4) \\
&\quad\times \zeta(z_2+z_3)\zeta(z_2+z_4)\zeta(z_3+z_4) \\
+&i\frac{(2\pi)^{2z_2-1}h^{**}(1-z_2)}{\pi^2\zeta(1-z_2+z_1+z_3+z_4)}\zeta(1-z_2+z_1)\zeta(1-z_2+z_3)\zeta(1-z_2+z_4) \\
&\quad\times \zeta(z_1+z_3)\zeta(z_1+z_4)\zeta(z_3+z_4) \\
+&i\frac{(2\pi)^{2z_3-1}h^{**}(1-z_3)}{\pi^2\zeta(1-z_3+z_1+z_2+z_4)}\zeta(1-z_3+z_1)\zeta(1-z_3+z_2)\zeta(1-z_3+z_4) \\
&\quad\times \zeta(z_1+z_2)\zeta(z_1+z_4)\zeta(z_2+z_4) \\
+&i\frac{(2\pi)^{2z_4-1}h^{**}(1-z_4)}{\pi^2\zeta(1-z_4+z_1+z_2+z_3)}\zeta(1-z_4+z_1)\zeta(1-z_4+z_2)\zeta(1-z_4+z_3) \\
&\quad\times \zeta(z_1+z_2)\zeta(z_1+z_3)\zeta(z_2+z_3).
\end{aligned} \tag{4.10}$$

We turn to $\mathcal{G}_1(z;h)$; note that we assume (4.4)–(4.5). We replace the Estermann zeta-functions involved in $\mathcal{G}_1$ by the absolutely convergent Dirichlet series which are implied by the functional equation. Then, after some rearrangement, we get

$$\mathcal{G}_1(z;h) = \left(\mathcal{G}_1^+ + \mathcal{G}_1^-\right)(z;h), \tag{4.11}$$

where

$$\begin{aligned}
&\mathcal{G}_1^{\pm}(z;h) = \zeta(z_1+z_2)\zeta(z_3+z_4)\\
&\times \sum_{m,n\ge 1} \frac{\sigma_{z_2-z_1}(m)\sigma_{z_4-z_3}(n)}{m^{\frac12(-z_1+z_2+z_3+z_4)}n^{\frac12(z_1+z_2-z_3+z_4)}}\mathcal{K}(\pm m,n;\psi_\pm).
\end{aligned} \tag{4.12}$$

Here

$$\begin{aligned}
\psi_+(x) = \frac{1}{\pi^4}\int_{(\beta)} &(x/2)^{2s+z_1+z_2+z_3+z_4-2}\\
&\times \Gamma(1-z_1-s)\Gamma(1-z_2-s)\Gamma(1-z_3-s)\Gamma(1-z_4-s)\\
&\times \Big(\cos\pi(s+\tfrac12(z_1+z_2))\cos\pi(s+\tfrac12(z_3+z_4))\\
&\qquad + \cos\tfrac12\pi(z_1-z_2)\cos\tfrac12\pi(z_3-z_4)\Big)h^{**}(s)ds,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
\psi_-(x) = -\frac{1}{\pi^4}\int_{(\beta)} &(x/2)^{2s+z_1+z_2+z_3+z_4-2}\\
&\times \Gamma(1-z_1-s)\Gamma(1-z_2-s)\Gamma(1-z_3-s)\Gamma(1-z_4-s)\\
&\times \Big(\cos\tfrac12\pi(z_1-z_2)\cos\pi(s+\tfrac12(z_3+z_4))\\
&\qquad + \cos\tfrac12\pi(z_3-z_4)\cos\pi(s+\tfrac12(z_1+z_2))\Big)h^{**}(s)ds.
\end{aligned} \tag{4.14}$$

Invoking Weil's bound for $S(m,n;l)$, we see readily that (3.4) implies that the triple sum in (4.12) converges absolutely in the domain

$$\begin{aligned}
\Big\{z:\ &\mathrm{Re}\,(z_1+z_2+z_3+z_4) > \frac52 - 2\beta,\\
&\qquad \mathrm{Re}\,z_1, \mathrm{Re}\,z_2, \mathrm{Re}\,z_3, \mathrm{Re}\,z_4 < -\beta\Big\},
\end{aligned} \tag{4.15}$$

with any $-\frac72 < \beta < -\frac54$. That is, (4.6), (4.10), (4.11) yield an analytic continuation of $\mathcal{G}(z;h)$ to (4.15). Note that (4.15) contains (4.4)–(4.5).

5. As remarked already, interesting situations occur when we take all z_j to the vicinity of $\mathcal{X}$, which is defined by (2.6). However, the domain (4.15) does not contain such points. This necessitates analytic continuation of $\mathcal{G}_1(z;h)$. It is accomplished via spectral expansions of $\mathcal{K}(\pm m,n;\psi_\pm)$. We shall consider first the expansion in the plus case, with greater details; the minus case is much simpler, and will be treated later very briefly. The analytic continuation itself will be established in the next section.

One may wish to appeal to the Kloosterman-Spectral sum formula due to Kuznetsov, a version of which is in Theorem 2.3 of [4]. However, our ψ_+ does not appear to satisfy the condition (2.4.6) postulated there, as x tends to $+\infty$.

Accordingly we shall instead follow the argument employed in the proof of the theorem.

To this end, let us introduce the Kloosterman-sum zeta-function:

$$Z_{m,n}(s) = (2\pi\sqrt{mn})^{2s-1}\sum_{l\ge1} l^{-2s}S(m,n;l). \tag{5.1}$$

We have, by definition,

$$\begin{aligned}\mathcal{K}(m,n;\psi_+) = \frac{1}{\pi^4}\int_{(\beta)} & Z_{m,n}(s+\tfrac12(z_1+z_2+z_3+z_4-1)) \\ & \times\Gamma(1-z_1-s)\Gamma(1-z_2-s)\Gamma(1-z_3-s)\Gamma(1-z_4-s) \\ & \times\Big(\cos\pi(s+\tfrac12(z_1+z_2))\cos\pi(s+\tfrac12(z_3+z_4)) \\ & \qquad +\cos\tfrac12\pi(z_1-z_2)\cos\tfrac12\pi(z_3-z_4)\Big)h^{**}(s)ds. \end{aligned}\tag{5.2}$$

Then we invoke Kuznetsov's spectral expansion of $Z_{m,n}(s)$ as given in Lemma 2.5 of [4]: For any $m,n>0$ and $\operatorname{Re}s>\frac34$

$$Z_{m,n}(s) = \left\{Z^{(d)}_{m,n}+Z^{(h)}_{m,n}+Z^{(c)}_{m,n}\right\}(s) - \frac{\delta_{m,n}}{2\pi}\frac{\Gamma(s)}{\Gamma(1-s)}, \tag{5.3}$$

where

$$\begin{aligned} Z^{(d)}_{m,n} &= \frac12\sin\pi s\sum_{j\ge1}\alpha_j t_j(m)t_j(n)\Gamma(s-\tfrac12+i\kappa_j)\Gamma(s-\tfrac12-i\kappa_j), \\ Z^{(h)}_{m,n} &= \sum_{k\ge1}(2k-1)q_{m,n}(k)\frac{\Gamma(k-1+s)}{\Gamma(k+1-s)}, \\ Z^{(c)}_{m,n} &= \frac{1}{2\pi}\sin\pi s\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}\Gamma(s-\tfrac12+ir)\Gamma(s-\tfrac12-ir)dr. \end{aligned}\tag{5.4}$$

We have

$$\mathcal{K}(m,n;\psi_+) = \left\{\mathcal{K}^{(d)}+\mathcal{K}^{(h)}+\mathcal{K}^{(c)}+\mathcal{K}^{(\delta)}\right\}(m,n;\psi_+), \tag{5.5}$$

with an obvious arrangement of terms.

We claim that

$$\mathcal{K}^{(d)}(m,n;\psi_+) = \sum_{j\ge1}\alpha_j t_j(m)t_j(n)\Psi_+(z;\kappa_j), \tag{5.6}$$

where

$$\begin{aligned}\Psi_+(z;r) = \frac{1}{2\pi^4}\int_{-i\infty}^{i\infty} & h^{**}(s)\Gamma(s-1+\tfrac12(z_1+z_2+z_3+z_4)+ir) \\ \times\,& \Gamma(s-1+\tfrac12(z_1+z_2+z_3+z_4)-ir)\Gamma(1-z_1-s)\Gamma(1-z_2-s) \\ \times\,& \Gamma(1-z_3-s)\Gamma(1-z_4-s)\sin\pi(s+\tfrac12(z_1+z_2+z_3+z_4-1)) \\ \times\,& \Big(\cos\pi(s+\tfrac12(z_1+z_2))\cos\pi(s+\tfrac12(z_3+z_4)) \\ & \qquad +\cos\tfrac12\pi(z_1-z_2)\cos\tfrac12\pi(z_3-z_4)\Big)ds. \end{aligned}\tag{5.7}$$

Here the path of integral is curved to ensure that the poles of the first three factors of the integrand lie to the left, and those of other factors to the right, respectively, while parameters are such that the path can be drawn.

This is easy to confirm. In fact, with $r \in \mathbb{R}$ and z in (4.15), we may take the vertical line (β) as the path. Then (3.4) and the Stirling formula implies readily that the integral of the absolute value of the integrand in (5.7) is $\ll_z (|r|+1)^{-4}$. Also we have the bounds (1.4) and

$$\sum_{\kappa_j \le K} \alpha_j \ll K^2 \tag{5.8}$$

(see (2.3.2) of [4]). By these facts, we see that (5.6) converges uniformly and absolutely in any compactum of (4.15). Obviously we have

$$\mathcal{K}^{(c)}(m,n;\psi_+) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}\Psi_+(z;r)dr, \tag{5.9}$$

as well. Incidentally we have proved that for z in (4.15) and $r \in \mathbb{R}$

$$\Psi_+(z;r) \ll (|r|+1)^{-4}. \tag{5.10}$$

Further, we have

$$\mathcal{K}^{(h)}(m,n;\psi_+) = \frac{2}{\pi}\sum_{k\ge 1}(-1)^k(2k-1)q_{m,n}(k)\Psi_+(z;i(\tfrac{1}{2}-k)). \tag{5.11}$$

This is a simple consequence of

$$q_{m,n}(k) \ll \frac{(2\pi\sqrt{mn})^{2k-1}}{\Gamma(2k)}\sum_{l\ge 1}|S(m,n;l)|l^{-2} \tag{5.12}$$

(see (2.2.7) of [4]) and, for z in (4.15),

$$\begin{aligned}\Psi_+(z;i(\tfrac{1}{2}-k)) = \frac{(-1)^k}{2\pi^3}\int_{(\beta)} h^{**}(s)&\frac{\Gamma(s+\frac{1}{2}(z_1+z_2+z_3+z_4-3)+k)}{\Gamma(k-s-\frac{1}{2}(z_1+z_2+z_3+z_4-3))}\\ \times\Gamma(1-z_1-s)&\Gamma(1-z_2-s)\Gamma(1-z_3-s)\Gamma(1-z_4-s)\\ \times\Big(\cos\pi(s&+\tfrac{1}{2}(z_1+z_2))\cos\pi(s+\tfrac{1}{2}(z_3+z_4))\\ &+\cos\tfrac{1}{2}\pi(z_1-z_2)\cos\tfrac{1}{2}\pi(z_3-z_4)\Big)ds.\end{aligned} \tag{5.13}$$

Corresponding to (5.10) we have

$$\Psi_+(z;i(\tfrac{1}{2}-k)) \ll k^{-4} \quad (k=1,2,3\ldots) \tag{5.14}$$

uniformly for any compactum of

$$\Big\{z : \frac{5}{2} - 2\beta < \mathrm{Re}\,(z_1 + z_2 + z_3 + z_4) < 6,\\ \mathrm{Re}\, z_1, \mathrm{Re}\, z_2, \mathrm{Re}\, z_3, \mathrm{Re}\, z_4 < -\beta\Big\}, \quad -\frac{7}{4} < \beta < -\frac{5}{4}. \tag{5.15}$$

To show this, we move the contour in (5.13) to the one which is the result of connecting, with straight lines, the points $\beta - i\infty$, $\beta - \frac{1}{2}ki$, $-4 - \frac{1}{2}ki$, $-4 + \frac{1}{2}ki$, $\beta + \frac{1}{2}ki$, and $\beta + i\infty$, in this order. We encounter only one singularity which is the simple pole of $h^{**}(s)$ at $s = -\frac{7}{2}$. The residue is $\ll k^{-10+\mathrm{Re}\,(z_1+z_2+z_3+z_4)}$. The integral over the new path is readily estimated to be $O(k^{-4})$ with (3.4) and Stirling's formula. Thus we have (5.14).

With this, we rearrange (5.11), following the argument on p. 66 of [4], and obtain, for z in (5.15),

$$\left(\mathcal{K}^{(h)} + \mathcal{K}^{(\delta)}\right)(m, n; \psi_+) = \sum_{k\ge 1} \sum_{j=1}^{\vartheta(k)} \alpha_{j,k} t_{j,k}(m) t_{j,k}(n) \Psi_+(z; i(\tfrac{1}{2} - k)), \tag{5.16}$$

with $\alpha_{j,k} = 16\Gamma(2k)(4\pi)^{-2k-1}|\varrho_{j,k}(1)|^2$. Note that we have

$$\sum_{j=1}^{\vartheta(k)} \alpha_{j,k} \ll k \tag{5.17}$$

(see (2.2.10) of [4]).

Collecting $(4.12)^+$, (5.5), (5.6), (5.9) and (5.16), we find that for z in (5.15)

$$\mathcal{G}_1^+(z; h) = \{\mathcal{H}_+ + \mathcal{H}_0 + \mathcal{C}\}\,(z^*; \Psi_+(z; \cdot)), \tag{5.18}$$

where for $z = (z_1, z_2, z_3, z_3)$

$$z^* = (z_1^*, z_2^*, z_3^*, z_4^*), \quad z_j^* = \frac{1}{2}(z_1 + z_2 + z_3 + z_4) - z_j; \tag{5.19}$$

$\mathcal{H}_+$, $\mathcal{C}$ are as in (2.3), but with an obvious extension of notation, and

$$\mathcal{H}_0(z; h) = \sum_{k\ge 1} \sum_{j=1}^{\vartheta(k)} \alpha_{j,k} H_{j,k}(z_1) H_{j,k}(z_2) H_{j,k}(z_3) H_{j,k}(z_4) h(i(\tfrac{1}{2} - k)). \tag{5.20}$$

The absolute convergence that is needed to carry out this rearranging procedure is guaranteed by the uniform bounds (1.4), (1.8) for Hecke eigenvalues, and by the bounds (5.8), (5.10), (5.14), (5.17).

We turn to $\mathcal{G}_1^-(z; h)$, or the spectral decomposition of $\mathcal{K}(-m, n; \psi_-)$. Note that z is now to be in the domain (5.15). This time we appeal to Theorem 2.5 of

[4]. The condition (2.4.6) of [4] is easily seen to be satisfied by $\psi_-(x)$, because of the exponential decay of the integrand in (4.14): According as $x \to +0$ and $+\infty$, we use the contours (β) and (-3), respectively.

Skipping the details, which are easy to fill, we have, corresponding to Ψ_+,

$$\begin{aligned}\Psi_-(z;r) = -\frac{\cosh \pi r}{2\pi^4} \int_{-i\infty}^{i\infty} h^{**}(s)\Gamma(s-1+\tfrac{1}{2}(z_1+z_2+z_3+z_4)+ir)\\ \times \Gamma(s-1+\tfrac{1}{2}(z_1+z_2+z_3+z_4)-ir)\Gamma(1-z_1-s)\Gamma(1-z_2-s)\\ \times \Gamma(1-z_3-s)\Gamma(1-z_4-s)\Big(\cos\tfrac{1}{2}\pi(z_1-z_2)\cos\pi(s+\tfrac{1}{2}(z_3+z_4))\\ +\cos\pi(s+\tfrac{1}{2}(z_1+z_2))\cos\tfrac{1}{2}\pi(z_3-z_4)\Big)ds,\end{aligned} \tag{5.21}$$

with the same path as in (5.7). Also we have, uniformly for bounded z in (5.15),

$$\Psi_-(z;r) \ll (|r|+1)^{-4}. \tag{5.22}$$

With this, we have, in much the same way as before,

$$\mathcal{G}_1^-(z;h) = \{\mathcal{H}_- + \mathcal{C}\}\,(z^*;\Psi_-(z;\cdot)). \tag{5.23}$$

Collecting (4.6), (4.11), (5.18), (5.23), we obtain, in the domain (5.15),

$$\begin{aligned}\mathcal{G}(z;h) = \mathcal{R}_1(z;h) + \mathcal{H}_+(z^*;\Psi_+(z;\cdot)) + \mathcal{H}_-(z^*;\Psi_-(z;\cdot))\\ + \mathcal{H}_0(z^*;\Psi_+(z;\cdot)) + \mathcal{C}(z^*;(\Psi_+ + \Psi_-)(z;\cdot)).\end{aligned} \tag{5.24}$$

6. We shall continue analytically the last expression for $\mathcal{G}(z;h)$. The article [2] lacks this decisive procedure altogether.

To this end, we introduce

$$\begin{aligned}\Xi_1(z;r) = \int_{-i\infty}^{i\infty} h^{**}(s)\frac{\Gamma\left(ir+\frac{1}{2}(z_1+z_2+z_3+z_4)-1+s\right)}{\Gamma\left(ir-\frac{1}{2}(z_1+z_2+z_3+z_4)+2-s\right)}\\ \times\Gamma(1-z_1-s)\Gamma(1-z_2-s)\Gamma(1-z_3-s)\Gamma(1-z_4-s)\\ \times\cos\pi\left(s+\tfrac{1}{2}(z_1+z_2)\right)\cos\pi\left(s+\tfrac{1}{2}(z_3+z_4)\right)ds,\end{aligned} \tag{6.1}$$

$$\begin{aligned}\Xi_2(z;r) = \int_{-i\infty}^{i\infty} h^{**}(s)\frac{\Gamma\left(ir+\frac{1}{2}(z_1+z_2+z_3+z_4)-1+s\right)}{\Gamma\left(ir-\frac{1}{2}(z_1+z_2+z_3+z_4)+2-s\right)}\\ \times\Gamma(1-z_1-s)\Gamma(1-z_2-s)\Gamma(1-z_3-s)\Gamma(1-z_4-s)ds,\end{aligned} \tag{6.2}$$

where the paths are the same and separate the poles of $h^{**}(s)\Gamma(ir+\frac{1}{2}(z_1+z_2+z_3+z_4)-1+s)$ to the left and those of $\Gamma(1-z_1-s)\Gamma(1-z_2-s)\Gamma(1-z_3-s)\Gamma(1-z_4-s)$ to the right; the $(z;r)\in\mathbb{C}^5$ is assumed to be such that the paths can be drawn. The integrand of (6.2) is of exponential decay, and the bound (3.4) assures amply the convergence of (6.1). Thus $\Xi_1(z;r)$ and $\Xi_2(z;r)$

are regular for those $(z;r)$ indicated. Shifting the paths appropriately we see also that they are meromorphic over $\mathbb{C}^5$; the polar divisors arise from those $(z;r)$ with which the path cannot be drawn.

In particular, if z is in (5.15) and $r \in \mathbb{R} \cup \{i(\frac{1}{2}-k) : k=1,2,3,\ldots\}$, then the line (β) can be used in (6.1) and (6.2). With this observation, we find, after some computation, that

$$\begin{aligned}\Psi_+(z;r) &= i\frac{\Xi_1(z;r)-\Xi_1(z;-r)}{4\pi^3\sinh\pi r}\\ &\quad + i\frac{\cos\frac{1}{2}\pi(z_1-z_2)\cos\frac{1}{2}\pi(z_3-z_4)}{4\pi^3\sinh\pi r}\left\{\Xi_2(z;r)-\Xi_2(z;-r)\right\}, \qquad (6.3)\\ \Psi_-(z;r) &= \frac{i}{4\pi^3\sinh\pi r}\Big(\cos\tfrac{1}{2}\pi(z_1-z_2)\cos\pi\left(\tfrac{1}{2}(z_1+z_2)+ir\right)\\ &\quad + \cos\tfrac{1}{2}\pi(z_3-z_4)\cos\pi\left(\tfrac{1}{2}(z_3+z_4)+ir\right)\Big)\Xi_2(z;r)\\ &\quad - \frac{i}{4\pi^3\sinh\pi r}\Big(\cos\tfrac{1}{2}\pi(z_1-z_2)\cos\pi\left(\tfrac{1}{2}(z_1+z_2)-ir\right)\\ &\quad + \cos\tfrac{1}{2}\pi(z_3-z_4)\cos\pi\left(\tfrac{1}{2}(z_3+z_4)-ir\right)\Big)\Xi_2(z;-r). \qquad (6.4)\end{aligned}$$

These identities obviously yield meromorphic continuations of $\Psi_\pm(z;r)$ to $\mathbb{C}^5$; that is, the two functions exist as far as the respective right sides are finite. Note that we have, for integers $k \geq 1$,

$$\begin{aligned}\Psi_+(z;i(\tfrac{1}{2}-k)) &= \frac{(-1)^k}{2\pi^3}\Xi_1(z;i(\tfrac{1}{2}-k))\\ &\quad + \frac{(-1)^k}{2\pi^3}\cos\tfrac{1}{2}\pi(z_1-z_2)\cos\tfrac{1}{2}\pi(z_3-z_4)\Xi_2(z;i(\tfrac{1}{2}-k)), \qquad (6.5)\end{aligned}$$

provided the right side is finite. Compare the above with Lemma 4.4 of [4].

We shall next estimate Ξ_1 and Ξ_2 as functions of r. To this end we introduce the set

$$\left\{z \in \mathbb{C}^4 : \text{ none of } z_j \text{ are equal to } \frac{9}{2}+a \text{ with an integer } a \geq 0\right\}. \qquad (6.6)$$

Note that this contains (5.15). Both Ξ_1 and Ξ_2 are regular in any compactum of (6.6), provided $|\mathrm{Re}\, r|$ is sufficiently large. Moreover, if r tends to infinity in any fixed horizontal strip, we have

$$\Xi_1(z;r) \ll |r|^{-4}e^{\pi|r|}, \qquad (6.7)$$

$$\Xi_2(z;r) \ll |r|^{-10+\mathrm{Re}\,(z_1+z_2+z_3+z_4)}, \qquad (6.8)$$

with implied constants depending only on $\max(|z_1|,|z_2|,|z_3|,|z_4|,|\mathrm{Im}\, r|)$. In fact, in this situation the paths in (6.1) and (6.2) can be drawn, and the regularity assertion is immediate. To prove the last bounds as $\mathrm{Re}\, r$ is positive

and large, we move the path to the one which is the result of connecting, with straight lines, the points $M-i\infty$, $M-\frac{1}{2}i|r|$, $-M-\frac{1}{2}i|r|$, $-M+i\infty$ in this order, where M is an integer such that $|r|>2M>8\max(|z_1|,|z_2|,|z_3|,|z_4|,|\mathrm{Im}\,r|)$. The singularities we encounter are only poles of $h^{**}(s)$. The residues contribute $\ll_M |r|^{-10+\mathrm{Re}\,(z_1+z_2+z_3+z_4)}$ to both Ξ_1 and Ξ_2. To estimate the integrals on the new path, we note that by (3.4) and Stirling's formula the integrand of Ξ_1 is estimated to be

$$\begin{aligned}&\ll_M |s+ir|^{\mathrm{Re}\,(s+\frac{1}{2}(z_1+z_2+z_3+z_4)+ir)-\frac{3}{2}}\\ &\times |s-ir|^{\mathrm{Re}\,(s+\frac{1}{2}(z_1+z_2+z_3+z_4)-ir)-\frac{3}{2}}|s|^{-\mathrm{Re}\,(2s+z_1+z_2+z_3+z_4)-2}e^{\pi|r|}, \quad (6.9)\end{aligned}$$

and that of Ξ_2 by the same expression but with $e^{-\pi|s|}$ in place of $e^{\pi|r|}$. The bounds (6.7)–(6.8) follow immediately.

Hence we have, uniformly for any compactum in (6.6),

$$\Psi_+(z;r) \ll |r|^{-4}, \quad (6.10)$$

$$\Psi_-(z;r) \ll |r|^{-10+\mathrm{Re}\,(z_1+z_2+z_3+z_4)}, \quad (6.11)$$

as r tends to infinity in any fixed horizontal strip. In much the same way one may show that uniformly for all integers $k\ge 1$ and for any compactum in (6.6)

$$\Psi_+(z;i(\tfrac{1}{2}-k)) \ll k^{-4}+k^{-10+\mathrm{Re}\,(z_1+z_2+z_3+z_4)}. \quad (6.12)$$

Now, the combination (6.10)–(6.12) implies an important assertion: the spectral sums $\mathcal{H}_\pm(z^*;\Psi_\pm(z;\cdot))$ and $\mathcal{H}_0(z^*;\Psi_+(z;\cdot))$ are all meromorphic in the domain

$$\prod_{j=1}^{4}\left\{z_j:\ \frac{1}{3}<\mathrm{Re}\,z_j<\frac{3}{2}\right\}. \quad (6.13)$$

To show this, we invoke the following large sieve estimates: For any fixed $\varepsilon>0$

$$\begin{aligned}&\sum_{\kappa_j\le K}\alpha_j|H_j(s)|^4,\ \sum_{k\le K}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}|H_{j,k}(s)|^4\\ &\qquad \ll_{\varepsilon,s} K^{2+8\max(0,\frac{1}{2}-\mathrm{Re}\,s)+\varepsilon}. \quad (6.14)\end{aligned}$$

Namely, it is enough to prove that for z in (6.13)

$$\sum_{j=1}^{4}\max\left(0,\tfrac{1}{2}-\mathrm{Re}\,z_j^*\right)<1, \quad (6.15)$$

which is, however, elementary.

We consider the contribution of the continuous spectrum in (5.24). Let z be in (5.15). By (6.3)–(6.4) we have

$$\mathcal{C}(z^*;(\Psi_++\Psi_-)(z;\cdot))=\mathcal{Y}(z;h), \quad (6.16)$$

where

$$\mathcal{Y}(z;h) = \frac{1}{\pi}\int_{(0)} \zeta(z_1^*+\xi)\zeta(z_1^*-\xi)\zeta(z_2^*+\xi)\zeta(z_2^*-\xi)$$
$$\times\, \zeta(z_3^*+\xi)\zeta(z_3^*-\xi)\zeta(z_4^*+\xi)\zeta(z_4^*-\xi)\frac{\Theta(z;-i\xi)}{\zeta(1+2\xi)\zeta(1-2\xi)}d\xi, \quad (6.17)$$

with

$$\Theta(z;r) = \frac{1}{2\pi^3\sinh\pi r}\Big[\Xi_1(z;r) + \Big\{\cos\pi(ir+\tfrac12(z_1+z_2))\cos\tfrac12\pi(z_1-z_2)$$
$$+\cos\pi(ir+\tfrac12(z_3+z_4))\cos\tfrac12\pi(z_3-z_4)$$
$$+\cos\tfrac12\pi(z_1-z_2)\cos\tfrac12\pi(z_3-z_4)\Big\}\Xi_2(z;r)\Big]. \quad (6.18)$$

Observe first that if z is in (5.15) then $\Xi_1(z;-i\xi)$ and $\Xi_2(z;-i\xi)$ are regular for $\operatorname{Re}\xi \ge -\frac54$. In fact, on noting the remark made after (6.6), possible singularities of these functions of ξ are at

$$-z_1^*-a_1,\ -z_2^*-a_2,\ -z_3^*-a_3,\ -z_4^*-a_4, \quad (6.19)$$

with non-negative integers a_j, and the real parts of these points are all less than $-\frac54$. We move the contour of the integral for $\mathcal{Y}$ to the one that is the result of connecting, with straight lines, the points $-i\infty$, $-Ni$, $[N]+\frac34-Ni$, $[N]+\frac34+Ni$, Ni, $+i\infty$, in this order. Here $N>0$ is large and such that $\zeta(s)\ne 0$ for $\operatorname{Im}s = 2N$. Assuming further that $|z_j|<\frac14 N$ $(1\le j\le 4)$, we have

$$\begin{aligned}\mathcal{Y}(z;h) =&[\text{the new integral}]\\ &+2i\times[\text{the sum of residues at } z_j^*-1\ (1\le j\le 4)]\\ &+2i\times[\text{the sum of residues at } \tfrac12\rho \text{ with } |\operatorname{Im}\rho|<2N],\end{aligned} \quad (6.20)$$

where ρ is a complex zero of $\zeta(s)$. Note that we have applied the functional equation for the zeta-function to $\zeta(1-2\xi)\sin\pi\xi$, and also that we have assumed that those relevant singularities are all simple poles, which obviously does not cause any loss of generality. This expression yields a meromorphic continuation of $\mathcal{Y}$ to the domain (6.13). In fact, the last two terms in (6.20) are meromorphic over $\mathbb{C}^4$; and the uniform convergence of the new integral for any compactum in (6.13) is a consequence of (6.7)–(6.8) and the easy bound

$$\int_0^T |\zeta(\sigma+it)|^8 dt \ll T^{2+8\max(0,\frac12-\sigma)}. \quad (6.21)$$

Thus $\mathcal{C}(z^*;(\Psi_+ + \Psi_-)(z;\cdot))$ exists as a meromorphic function in (6.13).

Observe that (5.15) and (6.13) are not disjoint. Namely, we have established the desired analytic continuation of the identity (5.24).

7. We now specialize the above by taking z close to the set $\mathcal{X}$. The functions $\Xi_1(z;r)$ and $\Xi_2(z;r)$ are regular in such z, for each $r \in \mathbb{R}$; one may use, for instance, the line $(\frac{1}{4})$ as the path in (6.1) and (6.2). This implies, via (6.3)–(6.4), $\Psi_\pm(z;r)$ are regular in the same way. Similarly, $\Psi_+(z; i(\frac{1}{2}-k))$ is regular. Hence $\mathcal{H}_\pm(z^*;\Psi_\pm(z;\cdot))$ and $\mathcal{H}_0(z^*;\Psi_+(z;\cdot))$ are all regular for the current specialization of z. As to $\mathcal{Y}(z;h)$, we shift the contour on the right of (6.20) back to the imaginary axis. We have immediately

$$\mathcal{C}(z^*;(\Psi_+ + \Psi_-)(z;\cdot)) = (\mathcal{C}_0 + \mathcal{R}_0)\,(z^*;(\Psi_+ + \Psi_-)(z;\cdot)). \tag{7.1}$$

Collecting (2.7), (5.24), and (7.1), we obtain:

Theorem. *Let z be either in $\mathcal{X}$ or close to it. Then we have the functional equation*

$$\begin{aligned}&(\mathcal{H}_+ + \mathcal{C}_0 + \mathcal{R}_0 - \mathcal{R}_1)\,(z;h)\\ &= \mathcal{H}_+(z^*;\Psi_+(z;\cdot)) + \mathcal{H}_-(z^*;\Psi_-(z;\cdot))\\ &+ \mathcal{H}_0(z^*;\Psi_+(z;\cdot)) + (\mathcal{C}_0+\mathcal{R}_0)(z^*;(\Psi_+ + \Psi_-)(z;\cdot)),\end{aligned} \tag{7.2}$$

with the conventions introduced above.

This is a corrected version of Theorem 15 of [2].

Concluding Remark. The device shown in Section 3 was sketched in Kuznetsov's letter (received on May 16, 1991), and was attributed to him in [3] by the present author, to whom the above proof is due. It should, however, be recorded, to avoid any confusion in the future, that a few variants of the device had been known to specialists since the publication of Murty's idea [5].

As a matter of fact, the device is by no means of absolute necessity in treating the spectral fourth moment of Hecke L-functions. For instance, in [1] it is replaced, though implicitly, by a series of truncation procedures. What is really indispensable is the perfection of analytic continuation, which in the present context is achieved only in Section 6. This aspect becomes apparent if one considers $\mathcal{H}_-(z;h)$ instead. There the device is irrelevant, and the whole argument is clearly built upon the analytic continuation of the spectral decomposition. It should perhaps be added that in [1] the continuation procedure is hidden in a use of the explicit formula for the binary additive divisor sum.

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